*Indian J. Phys.* **80** (3), 241-245 (2006)

# Electron-electron interactions and the electrical resistivity of aluminium at low temperatures

O N Awasthi* and V K Pundhir

Department of Physics, Regional Institute of Education, National Council of Educational Research & Training, Shyamla Hills, Bhopal-462 013, Madhya Pradesh, India

E-mail : omnarain_awasthi@yahoo.co.in

*Received 21 June 2005, accepted 19 December 2005*

**Abstract** : The electron-electron interactions in aluminium metal have been examined keeping in view of the recent developments. The contribution of the electron-electron Umklapp scattering processes in the electrical resistivity of aluminium at low temperatures has been evaluated using a simplified spherical Fermi surface model with isotropic transition probability. Our values of the electrical resistivity so obtained, compare fairly well with the experimental results.

**Keywords** : Electrical resistvity, e-e interactions, temperature dependence.

**PACS Nos.** : 34.80.-i, 72.10.Bg, 72.15.-v

## 1. Introduction

Since 1968, a revolution has taken place in the field of low-temperature electrical resistivity of simple metals. Recent technological advances in measurements have yielded unexpected experimental results, which are totally at variance with earlier accepted theories.

At liquid helium temperatures, the electrical resistivity of metals is governed mainly by electron-electron and electron-phonon interactions.

Garland and Bowers [1] measured the electrical resistivity of aluminium, indium, sodium and potassium as a function of temperature from 1.2 K to 4.2 K. Their results clearly indicated the presence of a $T^2$-term in the electrical resistivity of aluminium and indium.

Sodium and potassium, however, did not show a $T^2$-term. More refined later measurements [2,3] for the electrical resistivity of sodium and potassium in the temperature region 0.5 K to 4.2 K, have however, indicated that these metals also have a $T^2$-term. Extensive experimental work for aluminium [4–7] also shows a $T^2$-term. Further, e-e interactions and their contributions to the electrical resistvity in alkali metals and aluminium have appeared in the literature [8–12].

The study of electron-electron interactions started with the pioneering work of Landau and Pomeranchuk [13,14] and Baber [15], who showed that e-e scattering should contribute to $\rho(T)$, a term of the form

$$\rho_{ee}(T) = A_{ee}T^2 \qquad (1)$$

with the coefficient $A_{ee}$ being constant for a given metal. Here, $A_{ee}$ contains both normal ($A_{Nee}$) and umklapp ($A_{Uee}$) components. Although the actual scattering of electrons by other electrons is predominantly normal scattering, such scattering is so ineffective in contributing to $\rho(T)$ in aluminium that $A_{Uee}$ is predicted to dominate $A_{Nee}$.

$A_{ee}$ has been estimated for the simple metals like alkali metals and aluminium *etc.* mostly by the three groups. The initial pioneering work was done by Lawrence and Wilkins [16] who assumed that the only interaction between the electrons was the repulsive Coulomb interaction. They obtained the values of $A_{ee}$ for K and

*Corresponding Author

Na. A decade later, MacDonald, Taylor and Geldart [17] recalculated $A_{ee}$ using a more sophisticated procedure, a more realistic Fermi surface, and most importantly, a higher-order phonon-mediated interaction between electrons. This phonon-mediated interaction increases $A_{Ueе}$, but leaves $A_{Nee}$ nearly unchanged. Given the difficulty of these calculations, and their sensitivity both to the detailed shape of the Fermi surface and to phonon-mediated scattering, even these values are probably uncertain by a factor of 2.

Similar values to those of MacDonald, Taylor and Geldart [17] were obtained soon after by Awasthi [18] and Awasthi and Sathish [19] using a much less sophisticated procedure.

These workers did not separate the resistivity into normal and umklapp contributions; rather, they took the lattice periodicity into account by means of an overlap integral involving the Bloch wavefunctions of electrons with their momenta separated by reciprocal lattice vectors. Their values compare fairly well with the experimental results.

The three calculations just described all assumed that the electron distribution function is isotropic at low temperatures because $\rho(o)$ is dominated by impurity scattering. These calculations thus all yielded $A_{Ueе} >> A_{Nee}$.

In the light of the above background, we decided to examine the e-e interactions in aluminium and their contribution to its electrical resistivity at low temperatures.

## 2. Theoretical formulation

### *2.1. Electron-electron scattering :*

As is well known, in the isotropic limit, the normal e-e scattering does not contribute to the electrical resistivity because in such collisions, the charge and the momentum are conserved. Umklapp processes, however, impart momentum to the lattice as a whole and they can thus contribute to the electrical resistivity. The magnitude of this contribution is, however, greatly reduced due to the operation of the Pauli's exclusion principle.

### *2.2. The scattering potential :*

Although the free electrons of a metal have electrostatic Coulomb fields by which they repel one another, the field at large distances is screened out by collective motions of the electron gas, and there only remains a residual short-range interaction. The potential to be used in e-e scattering, as given by Bohm and Pines [20–22] is of the 'screened Coulomb' type with a screening distance of a few atomic dimensions. The form of the potential is as follows :

$$V(r) = (1/r).\exp(-k_c r) \text{ (atomic unit)}, \tag{2}$$

where $k_c$ is the screening parameter.

According to Pines [22] the value of $k_c$ can be determined from the relation :

$$\beta = k_c/k_f = 0.353\ r_s^{1/2} \tag{3}$$

with $r_s$ = radius of the Wigner-Seitz sphere and $k_f$ = Fermi wave vector.

The self-consistency of the method adopted by Pines to evaluate $k_c$, depends on $\beta$ being quite small. If we believe relation (3), we find that the screening length in a degenerate electron gas, is given by the relation :

$$1/k_c = 1.47 r_s^{1/2} \tag{4}$$

in Bohr units. This is about the same as the atomic radius in a typical metal. The effect of correlation is to decouple the electrostatic interactions between an atom and the others in the lattice, except for the few ions and electrons in their immediate neighborhood.

### *2.3. Scattering probability :*

In view of the presence of a screened Coulomb potential, electrons are deflected by each other and hence, a scattering occurs. Due to the change in the motion of electrons, there is a change in the wave vectors of the electrons and hence, change in energies. The scattering can be understood as a transition of electrons into new energy levels. The probability of scattering is equal to the probability of corresponding energy changes of electrons when they change to new energy levels.

Consider an electron in state $\boldsymbol{k}_1$ which is scattered into the state $\boldsymbol{k}_3$ by collisions with another electron in the state $\boldsymbol{k}_2$ which is itself being scattered into state $\boldsymbol{k}_4$. According to Pauli's exclusion principle, this scattering process can take place if the initial electronic states $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are filled and the final states $\boldsymbol{k}_3$ and $\boldsymbol{k}_4$ are empty. In collisions, both momentum as well as energy should be conserved.

If $f_{k1}$, $f_{k2}$, $f_{k3}$ and $f_{k4}$ are the Fermi functions of the electrons, the transition probability is given by the golden rule as :

$$T(\boldsymbol{k}_1, \boldsymbol{k}_2 \rightarrow \boldsymbol{k}_3, \boldsymbol{k}_4) = 2\pi/\hbar\ |<\boldsymbol{k}_3, \boldsymbol{k}_4|V(r)|\boldsymbol{k}_1, \boldsymbol{k}_2>|^2 f_{k1} f_{k2}\ (1 - f_{k3})\ (1 - f_{k4}) \times \delta(\epsilon_{k3} + \epsilon_{k4} - \epsilon_{k1} - \epsilon_{k2}). \tag{5}$$

The $\delta$ function in eq. (5) takes care of the energy conservation.

*2.4. Normal and Umklapp scattering :*

In a normal scattering, the momentum is conserved in the following way :

$$\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}_3 + \boldsymbol{k}_4. \tag{6}$$

In these collisions, the electrons donot lose momentum at all and hence, they cannot give rise to any contribution to electrical resistivity.

Now, let us introduce the effect of the lattice on the electrons. We know that the wave function for the electrons in the lattice is not plane waves but they are the Bloch functions defined as

$$\psi_k(\boldsymbol{r}) = \exp(i\boldsymbol{k}.\boldsymbol{r})u_k(\boldsymbol{r}), \tag{7}$$

with $u_k(\boldsymbol{r} + \boldsymbol{l}) = u_k(\boldsymbol{r})$; $\boldsymbol{l}$ = lattice vector.

Using Born approximation, the matrix elements of $V(r)$ between initial and final states can be written as :

$$\langle \boldsymbol{k}_3, \boldsymbol{k}_4|V(r)|\boldsymbol{k}_1,\boldsymbol{k}_2\rangle = \iint \Psi^*_{k3}(r_1)\Psi^*_{k4}(r_2)V(r) \times \Psi_{k1}(r_1)\Psi_{k2}(r_2)d^3r_1 d^3r_2. \tag{8}$$

Here, $\psi_k$ 's are the Bloch functions for the electrons in the lattice.

On introducing (7) for the Bloch functions into eq. (8) and simplifying it, we find that the matrix elements of $V(r)$ contain the square of an overlap integral $G$ given by :

$$G = 1/\Omega \int_{\text{Cell}} u^*_{k1}(\boldsymbol{r})u_{k3}(\boldsymbol{r})\exp(i\boldsymbol{g}.\boldsymbol{r})d^3r, \tag{9}$$

where $\boldsymbol{g} = \boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4$, $\boldsymbol{g}$ = reciprocal lattice vector, and $\Omega$ is the atomic volume.

The integral (9) is to be evaluated over a Wigner-Seitz cell. Proceeding in a similar manner as discussed by Ziman [23] and by Rösler [24], we obtain the following expression for $\rho_{ee}$ due to e-e Umklapp scattering processes :

$$\rho_{ee}(T) = (\pi^3 z/16)(e^2/v^*_F E^{*3}_F)k^2_B.(G^2/\beta^{*2})T^2 = A_{ee}T^2 \quad (\text{for } g \equiv 2k_F), \tag{10}$$

where $z$ = coordination number of the reciprocal lattice, $v_F^*$ = Fermi velocity (= $\hbar k_F/m^*$), $E_F^*$ = Fermi energy, $k_F$ = Fermi wave vector and $k_B$ is the Boltzmann constant.

Further, * (asterisk) on the symbols denotes that the value is evaluated with the effective mass of the electron in aluminium.

In deriving the electrical resistivity $\rho_{ee}(T)$ from eq. (10), certain assumptions were made.

Most important of them are : (a) Umklapp scattering is isotropic, and (b) the Fermi surface is spherical.

To test theoretical predictions with these assumptions in mind, one has to choose none other than simple metals like alkali metals and aluminium where their Fermi surfaces are believed to be nearly spherical in nature.

In the expression for $A_{ee}$ (eq.(10)), all the values are known except for $G$.

We now proceed to evaluate this important factor, which plays the crucial role in the e-e Umklapp scattering processes.

## 3. Evaluation of the overlap integral

In order to evaluate $G$ for aluminium, we need to have the knowledge of the wavefunctions of the electron in aluminium, which we have taken from the literature [25]. The wavefunctions for the simple metals are generally obtained by using the Wigner-Seitz's method [26,27]. This way, we find the wavefunction of an electron in the lowest state of the metal, *i.e.*, for $\boldsymbol{k} = 0$. Denoting the wavefunction in the ground state by $\psi_0(\boldsymbol{r})$, a fair approximation to the wave function for higher states will be (within any one atomic sphere), $\psi_k(\boldsymbol{r}) \equiv \exp(i\boldsymbol{k}.\boldsymbol{r})\psi_0(\boldsymbol{r})$, provided that $\boldsymbol{k}$ lies within the first Brillouin zone, but not too near the boundaries [28,29]. Wigner and Seitz have shown that this argument renders fairly accurate results for the case of fcc and bcc metals. In lattices with these structures, we can fill up the whole of space with polyhedra, one surrounding each atom. Near the boundary of each atomic polyhedron, the field will be small; near the middle, it will be spherically symmetrical.

To evaluate the overlap integral ($G$), eq. (9) is simplified. On normalization, $G$ can be written as

$$G = \frac{1}{g}\frac{\int_0^{r_s} |\psi_0(r)|^2 r\sin(gr)dr}{\int |\psi_0(r)|^2 r^2 dr} \tag{11}$$

where '$r_s$' is the radius of the Wigner-Seitz's sphere for the metal concerned. The value of $G$ for aluminium has been evaluated numerically using Simpson's rule. The range of $g$ has been taken from 0 to 2 in the units of $k_F$, the Fermi wave vector.

The variation of $G^2$ *vs.* $g$ for aluminium is shown in Figure 1.

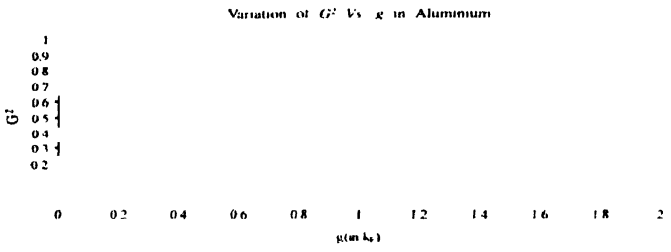


**Figure 1.** Variation of $G^2$ *vs.* $g$ in aluminium.

## 4. Evaluation of $A_{ee}$ for aluminium

In order to calculate the value of $A_{ee}$ for aluminium, we have to realize that aluminium crystallizes in the fcc structure, and its reciprocal lattice is bcc with coordination number $z = 8$, upto a reasonable approximation ($g = 2k_F$).

Since the square of the matrix element of $V(r)$ enters in eq. (5), it is the square of the function $G$ which largely determines the behaviour of the transition probability as a function of the change of electron momentum [30].

For e-e Umklapp scattering processes, the value of $g$ which is going to contribute for $\rho_{ee}$ must be $\cong 2k_F$, the Fermi diameter. The electrons which contribute to the conduction processes are usually those, which lie on or near Fermi surface.

The electrical conductivity of a metal depends directly on the area of the Fermi surface (among other factors). This Fermi area $A_F$ excludes the part covered by the zone boundaries, so that it measures only the 'free' surface area of the electron distribution. When the Fermi surface is spherical with the radius $k_F$ as for free electrons ($k_F = (3\pi^2 n)^{1/3}$), we have $A_F = 4\pi k_F^2$ , which is a function of the electron only. Except for monovalent metals, the true Fermi area must be rather less than this, because of the effect of zone boundaries. The effect of the band structure of the metallic crystal on the motion of the electron is usually accounted for by the introduction of a band effective mass $m^*$ of the electron. We therefore, look for a reasonable value of this effective mass $m^*$ of the electron for aluminium. The cyclotron resonance in aluminium has been observed by Langenberg and Moore [31] under anomalous skin conditions at 24 $GH_Z$.

They found an oscillation corresponding to an effective-mass $m^* = 1.5\ m$ ($m$ = free electron mass) with noticeable anisotropy. Electronic specific-heat measurements, on the other hand, give $m^* = 1.48$ m for aluminium.

In order to evaluate the value of $A_{ee}$ for aluminium, we have used $m^* = 1.5$m. Thus, if we put the effective mass in place of the free electron mass, the Fermi energy expression becomes

$$E_F^* = \hbar^2/2m^* \ (3\pi^2 n)^{2/3}. \tag{12}$$

Similarly,

$$\beta^* = 0.353(r_s)^{1/2}(m^*/m)^{1/2} = k_c/k_F. \tag{13}$$

## 5. Results

Our values of $A_{ee}$ for aluminium are listed in Table 1. For the sake of comparison, the experimental values from the literature [4–7] are also listed in Table 1. Our theoretical estimates are quite accurate and the probable error is less than 2%.

**Table 1.** $A_{ee}$ for aluminium. Comparison between experiment and theory $A_{ee}$ (pΩ cm $K^{-2}$).

| $A_{ee}$ (Experiment) | $A_{ee}$ (Theory) |
|---|---|
| 0.26 – 0.39 (±0.03) | 0.30 |

The experimental values of $A_{ee}$ reported in the literature vary from 0.26 pΩ cm $K^{-2}$ to 0.39 pΩ cm $K^{-2}$ in different samples of aluminium with varying experimental conditions at low temperatures.

## 6. Conclusion

From Table 1, it is obvious that our values of $A_{ee}$ for aluminium compare fairly well with the experimental data. It is concluded therefore, that our approach renders fairly good results for $\rho_{ee}(T)$ in aluminium at low temperatures.

### References

[1] J C Garland and R Bowers *Phys. Rev. Lett.* **21** 1007 (1968)

[2] B Levy, M Sinvani and A J Greenfield *Phys. Rev. Lett.* **43** 1822 (1979)

[3] J A Rowlands, C Duvvury and S B Woods *Phys. Rev. Lett.* **40** 1201 (1978)

[4] J C Garland and D J Van Harlingen *J. Phys.* **F8** 117 (1978)

[5] J H J M Ribot, J Bass, H Van Kempen and P Wyder *J. Phys.* **F9** L117 (1979)

[6] M Kaveh and N Wiser *Phys. Lett.* **51A** 89 (1975)

[7] M Sinvani, A J Greenfield, A Bergmann, M Kaveh and N Wiser *J. Phys.* **F11** 149 (1981)

[8] Jack Bass, William P Pratt, (Jr.) and Peter A Schroeder *Rev. Mod. Phys.* **62** 643 (1990)

[9] N Wiser *J. Phys. Cond. Matter (UK)* **4** L105 (1992)

[10] Y Z Hon *et al, J. Phys. Cond. Matter (UK)* **5** 7797 (1993)

[11] W W Schulz and P B Allen *Phys. Rev.* **B52** 7994 (1995)

[12] E Schafler and F Sachslehner *J. Phys. Cond. Matter (UK)* **12** 10499 (2000)

[13] L D Landau and I Pomeranchuk *Phys. Z. Sowjetunion* **10** 649 (1936)

[14] L D Landau and I Pomeranchuk *Zh. Eksp. Zh. Theo. Fiz.* **7** 379 (1937)

[15] W G Baber *Proc. Roy. Soc. (London)* **A158** 383 (1937)

[16] W E Lawrence and J W Wilkins *Phys. Rev.* **B7** 2317 (1973)

[17] A H MacDonald, R Taylor and D J W Geldart *Phys. Rev.* **B23** 2718 (1981)

[18] O N Awasthi *Nuovo Cim. Lett.* **15** 483 (1976)

[19] O N Awasthi and Sathish *Phys. Lett.* **A83** 283 (1981)

[20] D Bhom and D Pines *Phys. Rev.* **82** 625 (1951)

[21] D Bhom and D Pines *Phys. Rev.* **92** 609 (1953)

[22] D Pines *Solid State Phys.* **1** 106 (1955)

[23] J M Ziman *Electrons and Phonons* (Oxford : Clarendon) p412 (1960)

[24] M Rösler *Ann. Phys.* **16** 70 (1965)

[25] Segall Benjamin *Phys. Rev.* **124** 1797 (1961)

[26] E Wigner and F Seitz *Phys. Rev.* **43** 804 (1933)

[27] E Wigner and F Seitz *Phys.Rev.* **46** 509 (1934)

[28] N F Mott and H Jones *The Theory of the Properties of Metals and Alloys* (New York : Dover) p76 (1958)

[29] Charles Kittel: *Introduction to Solid State Physics* (7-th edn.) (New York : John Wiley) p248 (2003)

[30] Frank J. Blatt *Physics of Electronic Conduction in Solids* (New York : McGraw Hill) p152 (1968)

[31] D N Langenberg and T W Moore *Phys. Rev. Lett.* **3** 137 (1959)